Welcome..

A Unified Approach to Modal Reduction Methods

Part 1 of NASA SLaMS Series on Modal Synthesis

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Modal Synthesis

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Outline

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- This Webinar
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- Sets and Definitions
- Equivalence Theorem
- Fixed-Interface Methods
- Free-Interface Methods
- Mixed-Interface Methods
- Concluding Remarks
- Questions

Background

- The era of the modal reduction transformations began with Hurty (1965)
- The starting of the Space Shuttle Program spurred on tremendous R&D in modal reduction transformations and modal synthesis
- Today, even with our powerful 64-bit machines, multiple cores, and access to all the required RAM, modal reduction transformations and modal synthesis plays a key role in efficient system dynamic simulations
- Modal reduction/synthesis still remains a key and active R&D area in efficient computation/simulations … and you can contribute to this

Why Modal Synthesis? (1/2)

- With modal synthesis, you can break-up a large complex system into a “logical” set of components that make up the system
  - “logical” is often the actual interfaces between the components
- You can then use the appropriate modal transformations to significantly reduce the order of each component FEM
  - Component FEMs transform to component reduced-order Dynamic Math Models (DMMs)
- Use modal synthesis procedures to “synthesize” the component modes and calculate your system modes

Why Modal Synthesis? (2/2)

- Your original 5,000,000 DoF transient dynamics problem is now 500 DoFs with essentially no impact to accuracy
  - The speed of execution, numerical stability and convergence characteristics of your solution just increased by many orders of magnitude given the reduced-order system
    - You are now in in the driver-seat (instead of the FE-Solver)
  - You can easily and efficiently run all the sensitivity analyses that you can imagine to bound the problem and reduce mission risk
  - Would you have trusted a FE-solver with the solution of a 5,000,000 DoF transient dynamics problem to begin with?
- And there are times that you are the system integrator responsible for the system dynamic simulations
  - For this case, you will most likely be the receiver of DMMs from other organizations, so modal synthesis is required

Right Choice of Modal Reduction Method

- The answer is often dictated by the component's operating boundary conditions in the system configuration and available test data (not always aligned)
- This requires engineer familiarity with all modal reduction methods (fixed-, free-, and mixed- interface)
  - Often limited by the engineer's modal reduction knowledge/experience with most often opting for the method of most familiarity (Craig-Bampton)
- Component level test data (frequencies, mode shapes, damping) may also be available but often with different boundary conditions (free, over-constrained, …) than in operating system but can still be incorporated into the DMM formulation

Problem

- Engineer familiarity/comfort with methods often dictated by the complexity of formulation
- For example, Hurty's paper presents a fixed-interface method using a Ritz procedure that requires a special synthesis procedure
- MacNeal's free-interface method is built upon electrical engineering analogies
- Rubin's extension of MacNeal's method uses a truncated series expansion which does not lead to a transformation at all
- Herting's mixed interface method is a brute-force derivation
- Hintz, perhaps one of the most key figures in modal transformation methods, is a virtual unknown to most engineers given the complex nature of his paper, derivations, and special synthesis procedures
- All this has led to a general gravitation towards the common basis of the simplest method, Craig-Bampton, who wrote a very nice paper providing a simple transformation augmented with a realistic example problem

Our Objectives

- Present a simple systematic approach to the derivation of all modal reduction transformations
- Increase engineer familiarity/comfort level in using all methods as required
- Understand the strengths/weaknesses in each method
- Get to the point that you can build your own method that best suits your particular application and available test data

This Webinar

- This presentation is Part 1 of a two part NASA/SLaMs webinar series on Modal Synthesis
- Part 1 will cover a “Unified Approach to Modal Reduction Methods”
- Part 2 will cover
  - Modal synthesis methods
  - Response recovery methods
  - Special topics – Test Analytical Models
- Presentations are “Webinar” style; i.e., the slides require speaker commentary for completeness

Methods Covered

- Hurty
- Craig-Bampton
- Modified Hurty
- Modified Hurty w/ Attachment Modes
- Modified Hurty w/ flexibility
- Bamford
- Craig-Bampton with Inertia-Relief Modes
- Rubin
- MacNeal
- Craig-Chang
- Craig-Bampton w/ Free Interface Normal Modes
- Modified Craig-Bampton w/ Free Interface Normal Modes
- Modified Craig-Bampton w/ Free Interface Normal Modes and Inertia Relief
- Alternative Free Interface Method
- Hintz Method of Constraint Modes
- Herting
- Modified Hintz Method of Constraint Modes
- Hintz Method of Attachment Modes
- Modified Hintz Method of Attachment Modes
- Alternative Mixed Interface Methods

Set Definitions & Superscripts

Set	Size	Description
F	f	All FEM Physical DoFs
R	r	Statically Determinate Support DoFs
B	b	Redundant Interface DoFs
C	c	Additional redundant Interface DoFs
J	j	R + B
T	t	All Interface DoFs (T = R + B + C = J + C)
O	o	Interior DoFs (Complement of T in F)
K	k	Modal DoFs

Superscripts:

C Constraint Modes
R Rigid-body Modes; Residual Flexibility Modes
E Elastic Normal Modes
A Attachment Modes
N Rigid + Elastic Normal Modes

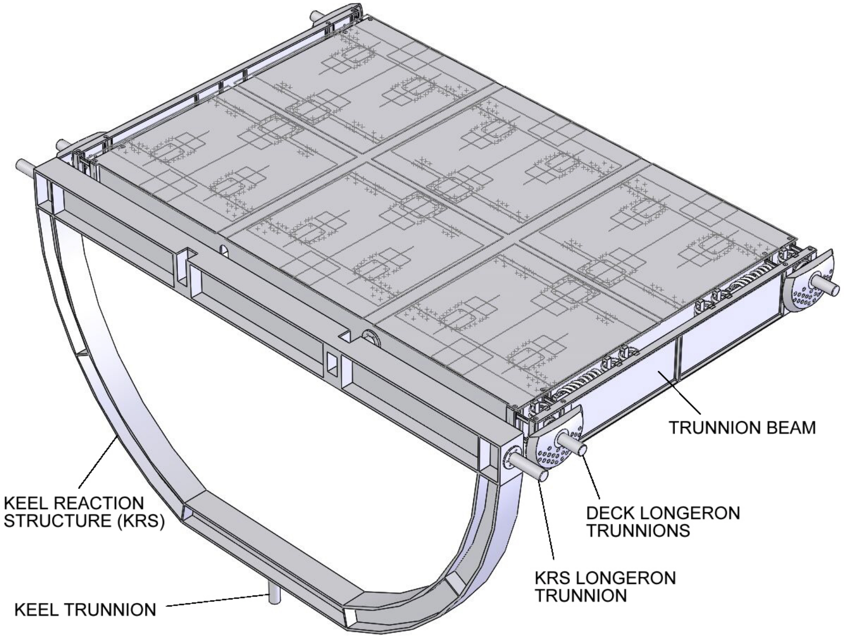
Equivalence Theorem: (Linear Algebra)

If $B=PAQ$, where P and Q are nonsingular matrices, then A and B are equivalent

- *Special case:*
  - *If $P = I$, $B = AQ$ then B and A are equivalent iff the inverse(Q) exists*

Fixed-Interface Methods (Def.)

ALL physical DoFs in the Modal Reduction Coordinate Transformation Must be FIXED when Calculating Normal Modes



Elastic Normal Modes

- Modal Solution

$$\left(K-\omega_k^2 M\right)\Phi_k^N=0$$

$$\Phi_k^T M \Phi_k = I_{kk}$$

$$\Phi_k^T K \Phi_k = \omega_k^2$$

$$\Phi_k^N=\begin{bmatrix}\Phi_{tk}^N\\ \Phi_{ok}^N\end{bmatrix}$$ Free-Interface

$$\Phi_k^N=\begin{bmatrix}0\\ \Phi_{ok}^N\end{bmatrix}$$ Fixed-Interface

$$\Phi_k^N=\begin{bmatrix}0\\ \Phi_{ck}^N\\ \Phi_{ok}^N\end{bmatrix}$$ Mixed-Interface

$$\left(\left(K+K_{bb}^{\beta}\right)-\omega_k^2\left(M+M_{bb}^{\beta}\right)\right)\Phi_k^N=0$$ Loaded-Interface Modes

Constraint Modes (T-set)

$$\begin{bmatrix} K_{tt} & K_{\mathrm{to}} \\ K_{ot} & K_{oo} \end{bmatrix} \begin{bmatrix} I_{tt} \\ \Phi_{ot}^{C} \end{bmatrix} = \begin{bmatrix} F_{tt} \\ 0 \end{bmatrix}$$

$$\Phi_{ot}^{C} = -K_{oo}^{-1} K_{ot}$$

Recall T = R + B

Constraint Modes (B-set)

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} 0 \\ I_{bb} \\ \Phi_{ob}^{C} \end{bmatrix} = \begin{bmatrix} F_{rb} \\ F_{bb} \\ 0 \end{bmatrix}$$

$$\Phi_{ob}^{C} = -K_{oo}^{-1} K_{ob}$$

Rigid-Body Modes
(R-set; Constraint Modes)

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} I_{rr} \\ \Phi^{R}_{\mathrm{br}} \\ \Phi^{R}_{\mathrm{or}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Phi^{R}_{\mathrm{br}} \\ \Phi^{R}_{\mathrm{or}} \end{bmatrix} = -\begin{bmatrix} K_{bb} & K_{bo} \\ K_{ob} & K_{oo} \end{bmatrix}^{-1} \begin{bmatrix} K_{br} \\ K_{\mathrm{or}} \end{bmatrix}$$

Hurty's Method (1965)

$$\Phi = \left[\Phi^R \, \Phi^C \, \Phi^E \right]$$

Hurty

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi^R_{br} & I_{bb} & 0 \\ \Phi^R_{\mathrm{or}} & \Phi^C_{\mathrm{ob}} & \Phi^E_{ok} \end{bmatrix} \begin{Bmatrix} x_r \\ q_b \\ q_k \end{Bmatrix}$$

Hurty (expanded out)

$$\bar{K} = \begin{bmatrix} K_{rr} & K_{rb} & K_{rk} \\ K_{br} & K_{bb} & K_{bk} \\ K_{kr} & K_{kb} & K_{kk} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{bb} & 0 \\ 0 & 0 & \omega_k^2 \end{bmatrix}$$

Special Properties

Craig-Bampton (1968)

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ q_b \\ q_k \end{Bmatrix}$$

Hurty

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ \Phi_{ot}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Craig-Bampton

Equivalent Transformations?

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{C} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Craig-Bampton (expanded out)

Modification of Hurty's Method

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ q_b \\ q_k \end{Bmatrix}$$

Hurty (expanded out)

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{bmatrix} I_{rr} & 0 & 0 \\ -\Phi_{br}^{R} & I_{bb} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Hurty x elementary transformation

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{C} \Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified-Hurty

Check for Equivalence

$$\begin{bmatrix} I_{rr} & 0 & 0 \\ -\Phi_{br}^{R} & I_{bb} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & I_{bb} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix}$$

Is elementary transformation invertible? Yes

$$\begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{C}\Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix}$$

Equivalent transformations

Original Hurty

Modified-Hurty

Equivalence between Modified-Hurty and Craig-Bampton

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{C}\Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified-Hurty

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{C} & \Phi_{\mathrm{ob}}^{C} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Craig-Bampton (Expanded)

$$\boxed{\Phi_{\mathrm{or}}^{C} = \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{C}\Phi_{br}^{R}}$$

Attachment Modes (B-set)

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} 0 \\ \Phi_{bb}^{A} \\ \Phi_{ob}^{A} \end{bmatrix} = \begin{bmatrix} F_{rb} \\ I_{bb} \\ 0_{ob} \end{bmatrix}$$

$$\Phi_{ob}^{A} = -K_{oo}^{-1} K_{ob} \Phi_{bb}^{A}$$

Attachment Modes (B-set) Relation to Flexibility Matrix

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} 0 \\ \Phi_{bb}^{A} \\ \Phi_{ob}^{A} \end{bmatrix} = \begin{bmatrix} F_{rb} \\ I_{bb} \\ 0_{ob} \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{bb}^{A} \\ \Phi_{ob}^{A} \end{bmatrix} = \begin{bmatrix} K_{bb} & K_{bo} \\ K_{ob} & K_{oo} \end{bmatrix}^{-1} \begin{bmatrix} I_{bb} \\ 0_{ob} \end{bmatrix} = \begin{bmatrix} G_{bb} & G_{bo} \\ G_{ob} & G_{oo} \end{bmatrix} \begin{bmatrix} I_{bb} \\ 0_{ob} \end{bmatrix}$$

$$G_{ob} = -K_{oo}^{-1} K_{ob} G_{bb}$$

$$\Phi_{ob}^{A} = -K_{oo}^{-1} K_{ob} \Phi_{bb}^{A}$$

$$\Phi_{fb}^{A} = \begin{bmatrix} 0_{rr} & 0_{rb} & 0_{ro} \\ 0_{br} & G_{bb} & G_{bo} \\ 0_{\mathrm{or}} & G_{ob} & G_{oo} \end{bmatrix} \begin{bmatrix} 0_{rb} \\ I_{bb} \\ 0_{ob} \end{bmatrix}$$

Attachment Modes (B-set) Relation to B-set Constraint Modes

- B-set Constraint Modes

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} 0 \\ I_{bb} \\ \Phi_{ob}^{C} \end{bmatrix} = \begin{bmatrix} F_{rb} \\ F_{bb} \\ 0_{ob} \end{bmatrix}$$

$$\Phi_{ob}^{C} = -K_{oo}^{-1} K_{ob}$$

- B-set Attachment Modes

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} 0 \\ \Phi_{bb}^{A} \\ \Phi_{ob}^{A} \end{bmatrix} = \begin{bmatrix} F_{rb} \\ I_{bb} \\ 0_{ob} \end{bmatrix}$$

$$\Phi_{ob}^{A} = -K_{oo}^{-1} K_{ob} \Phi_{bb}^{A}$$

$$\boxed{\Phi_{ob}^{A} = \Phi_{ob}^{C} \Phi_{bb}^{A}}$$

For this case (B-set), attachment modes are a linear combination of constraint modes

What this means is that we can build an equivalent Hurty's Method with B-set Attachment Modes

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & \Phi_{bb}^{A} & 0 \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{A} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ q_b \\ q_k \end{Bmatrix}$$

Construct Hurty's Method W/ B-set Attachment Modes

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi_{br}^{R} & \Phi_{bb}^{A} & 0 \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{A} & \Phi_{ok}^{E} \end{bmatrix} \begin{bmatrix} I_{rr} & 0 & 0 \\ -\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \left(\Phi_{bb}^{A}\right)^{-1} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Post-Multiply by elementary transformation

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{A}\left(\Phi_{bb}^{A}\right)^{-1} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified Hurty's Method w/ B-set Attachment Modes

Now we have TWO Equivalent and Useful Forms of the Hurty Transformation

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi^R_{\mathrm{or}} - \Phi^C_{ob}\Phi^R_{br} & \Phi^C_{\mathrm{ob}} & \Phi^E_{ok} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified-Hurty w/ Constraint Modes

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi^C_{\mathrm{or}} & \Phi^C_{\mathrm{ob}} & \Phi^E_{ok} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Craig-Bampton (Expanded)

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi^R_{\mathrm{or}} - \Phi^A_{ob}\left(\Phi^A_{bb}\right)^{-1}\Phi^R_{br} & \Phi^A_{\mathrm{ob}}\left(\Phi^A_{bb}\right)^{-1} & \Phi^E_{ok} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified-Hurty w/ Attachment Modes

Hurty's Method w/ Attachment Modes in Terms of Flexibility

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{A}\left(\Phi_{bb}^{A}\right)^{-1} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified Hurty's Method w/ Attachment Modes

$$\Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1} = G_{ob} G_{bb}^{-1}$$

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ 0 & I_{bb} & 0 \\ \Phi_{\mathrm{or}}^{R} - G_{ob} G_{bb}^{-1}\Phi_{br}^{R} & -G_{ob} G_{bb}^{-1} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_k \end{Bmatrix}$$

Modified Hurty's Method in Terms of Flexibility

Bamford (1966) (NASA/JPL) Method of Attachment Modes Constrained Components Only

$$\begin{Bmatrix} x_r \\ \boxed{\begin{matrix} x_b \\ x_o \end{matrix}} \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 \\ \Phi^R_{br} & \Phi^A_{bb} & 0 \\ \Phi^R_{\mathrm{or}} & \Phi^A_{\mathrm{ob}} & \Phi^E_{ok} \end{bmatrix} \begin{Bmatrix} x_r \\ \boxed{\begin{matrix} q_b \\ q_k \end{matrix}} \end{Bmatrix}$$

Take our Hurty's method w/ Attachment Modes and partition out the rigid-body

$$\begin{Bmatrix} x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} \Phi^A_{bb} & 0 \\ \Phi^A_{\mathrm{ob}} & \Phi^E_{ok} \end{bmatrix} \begin{Bmatrix} q_b \\ q_k \end{Bmatrix}$$

Bamford's Method (Constrained Components Only)

Bamford introduced the B-set type attachment modes and developed a modal reduction coordinate transformation applicable for constrained components. The method required a special synthesis procedure which can be mitigated with elementary transformation.

Inertia-Relief Attachment Modes (R-set)

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Phi_{\mathrm{or}}^{A} \end{bmatrix} = -\begin{bmatrix} M_{rr} & M_{rb} & M_{ro} \\ M_{br} & M_{bb} & M_{bo} \\ M_{\mathrm{or}} & M_{ob} & M_{oo} \end{bmatrix} \begin{bmatrix} I_{rr} \\ \Phi_{br}^{R} \\ \Phi_{\mathrm{or}}^{R} \end{bmatrix} + \begin{bmatrix} F_{rr} \\ F_{br} \\ 0 \end{bmatrix}$$

$$\Phi_{\mathrm{or}}^{A} = -K_{oo}^{-1} \left(M_{oo} \Phi_{\mathrm{or}}^{R} + M_{ob} \Phi_{br}^{R} + M_{\mathrm{or}} \right)$$

Craig-Bampton w/ Inertia-Relief Attachment Modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ \Phi_{ot}^C & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Original Craig-Bampton

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^C & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Craig-Bampton w/ Inertia-Relief Modes

Potential for linear dependency

Special Topic: Over-Constrained Craig-Bampton Damping Matrix

Problem: You have an over-constrained C-B stiffness and mass matrix. You also have modal damping values from a nominally constrained test configuration. You know leaving the over-constrained physical partition undamped can lead to big problems in the CLA. What can you do?

$$\begin{Bmatrix} x_b \\ x_c \\ q_k \end{Bmatrix} = \begin{bmatrix} I_{bb} & 0 \\ \Phi^{C}_{\mathrm{c+k,b}} & \Phi^{E}_{\mathrm{c+k,c+k}} \end{bmatrix} \begin{Bmatrix} x_b \\ q_{\mathrm{c+k}} \end{Bmatrix} = T \begin{Bmatrix} x_b \\ q_{\mathrm{c+k}} \end{Bmatrix}$$

Note this is not
a reduction of coordinates

$$T^{T} C\, T = \begin{bmatrix} 0_{bb} & 0_{\mathrm{b,c+k}} \\ 0_{\mathrm{c+k,b}} & 2\zeta_{\mathrm{c+k}}\,\omega_{\mathrm{c+k}} \end{bmatrix}$$

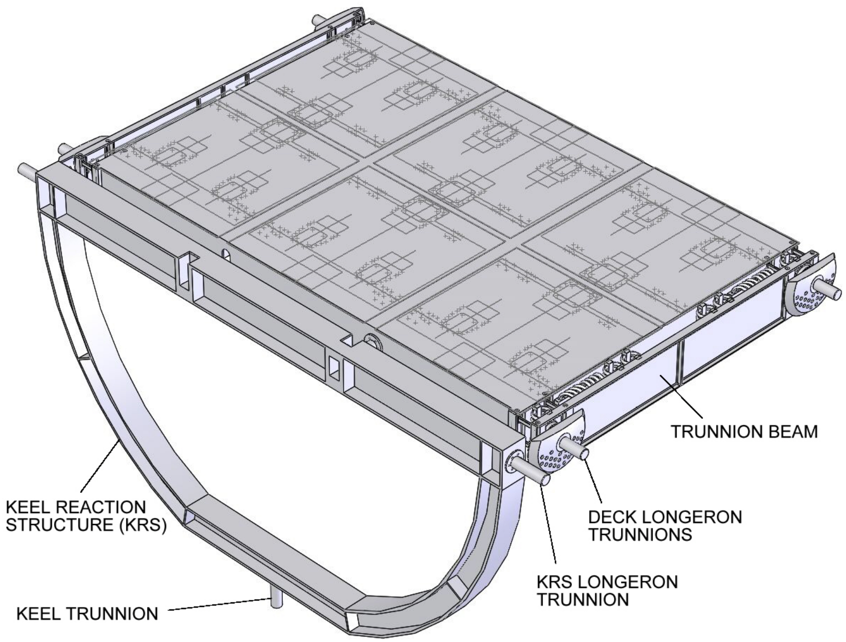
We know that the triple product of the unknown over-constrained damping matrix with the subject transformation is the nominally constrained damping matrix

$$C = \left(T^{T}\right)^{-1} \begin{bmatrix} 0_{bb} & 0_{\mathrm{b,c+k}} \\ 0_{\mathrm{c+k,b}} & 2\zeta_{\mathrm{c+k}}\,\omega_{\mathrm{c+k}} \end{bmatrix} T^{-1}$$

Solve for the over-constrained damping matrix

Free-Interface Methods (Def.)

<u>ALL</u> physical DoFs in the Modal Reduction Coordinate Transformation Must Be <u>FREE</u> when Calculating the Normal Modes



Inertia-Relief Attachment Modes (T-set)

$$M\ddot{x}+Kx=F_{ft}=\begin{bmatrix}I_{tt}\\0\end{bmatrix}$$

$$x=x_r+x_e$$

$$M\ddot{x}_e+Kx_e=F_{ft}-M\ddot{x}_r$$

$$M\ddot{x}_e+Kx_e=F_{ft}-M\Phi_r M_{rr}^{-1}\Phi_r^T F_{ft}$$

$$M\ddot{x}_e+Kx_e=\left(I-M\Phi_r M_{rr}^{-1}\Phi_r^T\right)F_{ft}$$

$$M\ddot{x}_e+Kx_e=AF_{ft}$$

Inertia-Relief Attachment Modes (T-set)

$$M\,\ddot{x}_e + K\,x_e = A\,F_{ft}$$

$$\begin{bmatrix} K_{rr} & K_{rb} & K_{ro} \\ K_{br} & K_{bb} & K_{bo} \\ K_{\mathrm{or}} & K_{ob} & K_{oo} \end{bmatrix}\begin{bmatrix} 0 \\ \Phi_{bt}^{A} \\ \Phi_{ot}^{A} \end{bmatrix} = \begin{bmatrix} A_{rr} & A_{rb} & A_{ro} \\ A_{br} & A_{bb} & A_{bo} \\ A_{\mathrm{or}} & A_{ob} & A_{oo} \end{bmatrix} F_{ft} + \begin{bmatrix} R_{rt} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \Phi_{bt}^{A} \\ \Phi_{ot}^{A} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & G_{bb} & G_{bo} \\ 0 & G_{ob} & G_{oo} \end{bmatrix}\begin{bmatrix} A_{rr} & A_{rb} & A_{ro} \\ A_{br} & A_{bb} & A_{bo} \\ A_{\mathrm{or}} & A_{ob} & A_{oo} \end{bmatrix} F_{ft}$$

$$\widetilde{\Phi}_{ft}^{A} = G_{ff}\,A_{ff}\,F_{ft}$$

Inertia-Relief Attachment Modes (T-set); Re-orthogonalize Relative to Rigid Body Modes

$$\widetilde{\Phi}_{ft}^{A} = G_{ff} A_{ff} F_{ft}$$

$$\Phi_{ft}^{A} = \widetilde{\Phi}_{ft}^{A} - \Phi_{fr}^{R} Q_{rt}$$

$$\left(\Phi_{fr}^{R}\right)^{T} M_{ff} \Phi_{ft}^{A} = 0$$

$$Q_{rt} = M_{rr}^{-1} \left(\Phi_{fr}^{R}\right)^{T} M_{ff} \widetilde{\Phi}_{ft}^{A}$$

$$\Phi_{ft}^{A} = \widetilde{\Phi}_{ft}^{A} - \Phi_{fr}^{R} M_{rr}^{-1} \left(\Phi_{fr}^{R}\right)^{T} M_{ff} \widetilde{\Phi}_{ft}^{A}$$

$$\Phi_{ft}^{A} = A_{ff}^{T} \widetilde{\Phi}_{ft}^{A}$$

Inertia-Relief Attachment Modes (T-set); Final Steps

$$\tilde{\Phi}_{ft}^{A} = G_{ff} A_{ff} F_{ft}$$

$$\Phi_{ft}^{A} = A_{ff}^{T} \tilde{\Phi}_{ft}^{A}$$

$$\Phi_{ft}^{A} = \left(A_{ff}^{T} G_{ff} A_{ff} \right) F_{ft}$$

$$\boxed{\Phi_{ft}^{A} = G_{ff}^{A} F_{ft}}$$

Inertia-Relief Residual Flexibility Modes (T-set)

$$\Phi_{ft}^{R} = \left(G_{ff}^{A} - \Phi_{fk}^{E} \omega_{kk}^{-2} \left(\Phi_{fk}^{E} \right)^{T} \right) F_{ft}$$

$$\boxed{\Phi_{ft}^{R} = G_{ff}^{R} F_{ft}}$$

Rubin (1975)

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G^R_{ot}\left(G^R_{tt}\right)^{-1} & \Phi^N_{ok} - G^R_{ot}\left(G^R_{tt}\right)^{-1}\Phi^N_{tk} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Rockwell International*
Late 70's early 80's in
a complex ad-hoc
derivation

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G^R_{tt} & \Phi^N_{tk} \\ G^R_{ot} & \Phi^N_{ok} \end{bmatrix} \begin{Bmatrix} q_t \\ q_k \end{Bmatrix}$$

Step 1 – Break it down
To component Modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G^R_{tt} & \Phi^N_{tk} \\ G^R_{ot} & \Phi^N_{ok} \end{bmatrix} \begin{bmatrix} \left(G^R_{tt}\right)^{-1} & -\left(G^R_{tt}\right)^{-1}\Phi^N_{tk} \\ 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Step 2 – Multiply
by elementary
transformation

* Henkel & Martens
Rockwell International
~late 70s

Rubin (Cont'd)

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R \left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R \left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

We get Rubin's Transformation

$$\begin{bmatrix} \left(G_{tt}^R\right)^{-1} & -\left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \\ 0 & I_{kk} \end{bmatrix} \begin{bmatrix} G_{tt}^R & \Phi_{tk}^N \\ 0 & I_{kk} \end{bmatrix} = I$$

Check for equivalence

$$\begin{bmatrix} \bar{K}_{tt} & K_{\mathrm{tk}} \\ K_{kt} & K_{kk} \end{bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R \left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R \left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \end{bmatrix}^T \begin{bmatrix} K_{tt} & K_{\mathrm{to}} \\ K_{ot} & K_{oo} \end{bmatrix} \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R \left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R \left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \end{bmatrix}$$

$$\begin{bmatrix} \bar{M}_{tt} & M_{\mathrm{tk}} \\ M_{kt} & M_{kk} \end{bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R \left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R \left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \end{bmatrix}^T \begin{bmatrix} M_{tt} & M_{\mathrm{to}} \\ M_{ot} & M_{oo} \end{bmatrix} \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R \left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R \left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \end{bmatrix}$$

Rubin Reduced Stiffness & Mass

MacNeal (1971)

$$\begin{bmatrix} \bar{K}_{tt} & K_{\mathrm{tk}} \\ K_{kt} & K_{kk} \end{bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^{R}\left(G_{tt}^{R}\right)^{-1} & \Phi_{ok}^{N} - G_{ot}^{R}\left(G_{tt}^{R}\right)^{-1}\Phi_{tk}^{N} \end{bmatrix}^{T} \begin{bmatrix} K_{tt} & K_{\mathrm{to}} \\ K_{ot} & K_{oo} \end{bmatrix} \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^{R}\left(G_{tt}^{R}\right)^{-1} & \Phi_{ok}^{N} - G_{ot}^{R}\left(G_{tt}^{R}\right)^{-1}\Phi_{tk}^{N} \end{bmatrix}$$

$$\begin{bmatrix} \bar{M}_{tt} & M_{\mathrm{tk}} \\ M_{kt} & M_{kk} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I_{kk} \end{bmatrix}$$

MacNeal Reduced Stiffness & Mass

Craig-Chang (1977)

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G_{tt}^R & \Phi_{tk}^N \\ G_{ot}^R & \Phi_{ok}^N \end{bmatrix} \begin{Bmatrix} q_t \\ q_k \end{Bmatrix}$$

What is the relationship between Craig-Chang and Rubin Transformations?

Relationship Between Craig-Chang and Rubin

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G_{tt}^R & \Phi_{tk}^N \\ G_{ot}^R & \Phi_{ok}^N \end{bmatrix} \begin{Bmatrix} q_t \\ q_k \end{Bmatrix}$$

Craig-Chang

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G_{tt}^R & \Phi_{tk}^N \\ G_{ot}^R & \Phi_{ok}^N \end{bmatrix} \begin{bmatrix} \left(G_{tt}^R\right)^{-1} & -\left(G_{tt}^R\right)^{-1}\Phi_{tk}^N \\ 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Craig-Chang x elementary transformation

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R\left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R\left(G_{tt}^R\right)^{-1}\Phi_{tk}^N \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Results in Rubin

$$\begin{bmatrix} \left(G_{tt}^R\right)^{-1} & -\left(G_{tt}^R\right)^{-1}\Phi_{tk}^N \\ 0 & I_{kk} \end{bmatrix} \begin{bmatrix} G_{tt}^R & \Phi_{tk}^N \\ 0 & I_{kk} \end{bmatrix} = I$$

Check equivalence (all information is retained)

Craig-Chang and Rubin Transformations are Equivalent

Craig-Bampton with Free-Interface Normal Modes?

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ \Phi_{ot}^C & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Original Craig-Bampton

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & \Phi_{tk}^E \\ \Phi_{ot}^C & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} q_t \\ q_k \end{Bmatrix}$$

Craig-Bampton w/
Free-Interface Normal Modes

Potential Linear Dependence!

Special Synthesis Procedure

Constructing Modified Craig-Bampton w/ Free-Interface Normal Modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & \Phi_{tk}^E \\ \Phi_{ot}^C & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} q_t \\ q_k \end{Bmatrix}$$

C-B w/ Free-Interface Modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & \Phi_{tk}^E \\ \Phi_{ot}^C & \Phi_{ok}^E \end{bmatrix} \begin{bmatrix} I_{tt} & -\Phi_{tk}^E \\ 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

x elementary transformation

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ \Phi_{ot}^C & \Phi_{ok}^E - \Phi_{ot}^C \Phi_{tk}^E \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Modified C-B w/ Free-Interface Modes

$$\begin{bmatrix} I_{tt} & -\Phi_{tk}^E \\ 0 & I_{kk} \end{bmatrix} \begin{bmatrix} I_{tt} & \Phi_{tk}^E \\ 0 & I_{kk} \end{bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ 0 & I_{kk} \end{bmatrix}$$

Check equivalence

Add Inertia-Relief Modes to Improve Accuracy

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Modified Craig-Bampton w/ Free-Interface Normal Modes
plus inertia-relief modes

Relationship Between Rubin and Modified Craig-Bampton w/ Free-Interface Normal Modes?

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^R \left(G_{tt}^R\right)^{-1} & \Phi_{ok}^N - G_{ot}^R \left(G_{tt}^R\right)^{-1} \Phi_{tk}^N \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Rubin

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^C & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E - \Phi_{ot}^C \Phi_{tk}^E \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Modified Craig-Bampton w/ Free-Interface
Normal Modes + Inertia Relief

Alternative Free-Interface* Inertia-Relief Total Flexibility Form

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G_{tt}^{A} & \Phi_{tk}^{N} \\ G_{ot}^{A} & \Phi_{ok}^{N} \end{bmatrix} \begin{Bmatrix} q_t \\ q_k \end{Bmatrix}$$

T-set <u>Total</u> Flexibility plus free-interface normal modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} G_{tt}^{A} & \Phi_{tk}^{N} \\ G_{ot}^{A} & \Phi_{ok}^{N} \end{bmatrix} \begin{bmatrix} \left(G_{tt}^{A}\right)^{-1} & -\left(G_{tt}^{A}\right)^{-1}\Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Post-multiply by elementary transformation

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^{A}\left(G_{tt}^{A}\right)^{-1} & \Phi_{ok}^{N} - G_{ot}^{A}\left(G_{tt}^{A}\right)^{-1}\Phi_{tk}^{N} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Final Transformation

$$\begin{bmatrix} \left(G_{tt}^{A}\right)^{-1} & -\left(G_{tt}^{A}\right)^{-1}\Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix} \begin{bmatrix} G_{tt}^{A} & \Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix} = I$$

Check for Equivalence

* Majed, Rockwell International, 1989

Method should be equivalent to Rubin

3 Equivalent Free-Interface Methods

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^{R}\left(G_{tt}^{R}\right)^{-1} & \Phi_{ok}^{N} - G_{ot}^{R}\left(G_{tt}^{R}\right)^{-1}\Phi_{tk}^{N} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Rubin

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix} \begin{pmatrix} x_t \\ q_r^{A} \\ q_k \end{pmatrix}$$

Modified Craig-Bampton With Free-Interface Normal Modes and Inertia Relief

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 \\ G_{ot}^{A}\left(G_{tt}^{A}\right)^{-1} & \Phi_{ok}^{N} - G_{ot}^{A}\left(G_{tt}^{A}\right)^{-1}\Phi_{tk}^{N} \end{bmatrix} \begin{Bmatrix} x_t \\ q_k \end{Bmatrix}$$

Inertia-Relief Total Flexibility Form

Preference may be based on ease of computation and/or programming
However, there are other considerations: experimental determination

Factor the Resulting Reduced Stiffness for Different Forms & Investigate

$$K^{\text{Rubin}} = \begin{bmatrix} \left(G_{tt}^{R}\right)^{-1} & -\left(G_{tt}^{R}\right)^{-1}\Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix}^{T} \begin{bmatrix} G_{tt}^{R} & 0 \\ 0 & \omega_{k}^{2} \end{bmatrix} \begin{bmatrix} \left(G_{tt}^{R}\right)^{-1} & -\left(G_{tt}^{R}\right)^{-1}\Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix}$$

$$K^{\text{Modified CB}} = \begin{bmatrix} I_{tt} & 0 & -\Phi_{tk}^{E} \\ 0 & I_{rr} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix}^{T} \begin{bmatrix} K_{tt}^{C} & 0 & K_{tk}^{CE} \\ 0 & K_{rr}^{A} & 0 \\ K_{kt}^{EC} & 0 & \omega_{k}^{2} \end{bmatrix} \begin{bmatrix} I_{tt} & 0 & -\Phi_{tk}^{E} \\ 0 & I_{rr} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix}$$

$$K^{\text{Alternate}} = \begin{bmatrix} \left(G_{tt}^{A}\right)^{-1} & -\left(G_{tt}^{A}\right)^{-1}\Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix}^{T} \begin{bmatrix} K_{tt}^{A} & K_{tk}^{AN} \\ K_{kt}^{NA} & \omega_{k}^{2} \end{bmatrix} \begin{bmatrix} \left(G_{tt}^{A}\right)^{-1} & -\left(G_{tt}^{A}\right)^{-1}\Phi_{tk}^{N} \\ 0 & I_{kk} \end{bmatrix}$$

All Equivalent; However, Rubin provides the superior form relative to test

Special Topic: Deriving Fixed-Interface Normal Modes from Free-Interface Normal Modes and Residual Flexibility

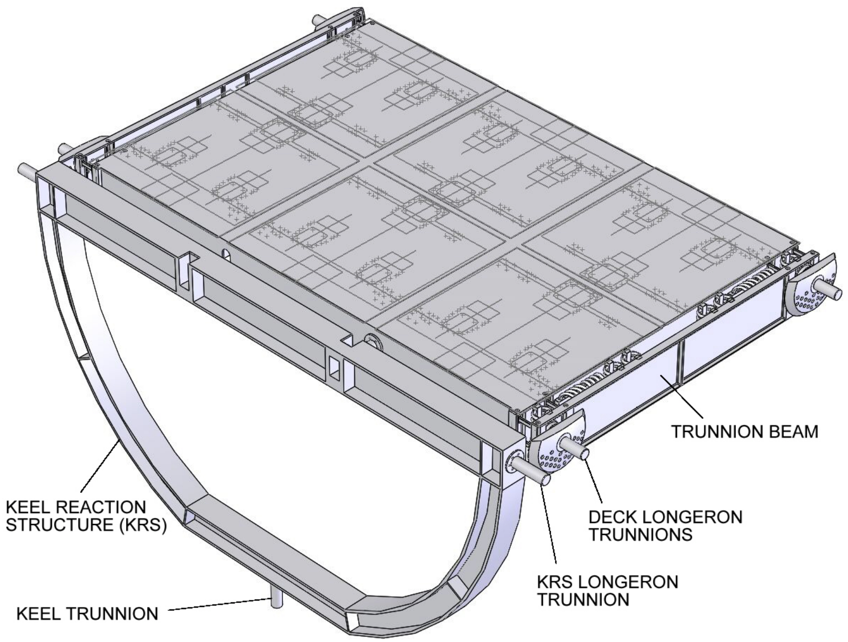
Why? All about comfort level of engineers with Craig-Bampton vs Rubin-MacNeal. Reason for doing this may be that you have executed an unconstrained modal test but would like to directly incorporate that data into a Craig-Bampton form (instead of Rubin-MacNeal). In the '90s, a journal publications came out on deriving fixed-interface modes from mass-loaded interface modes. Subsequent to that, another paper with the more robust method of deriving fixed-interface modes from free-interface modes and residual flexibility. The issue of deriving constraint modes was An open question. However, some of the relations derived in this presentation may be useful in that.

$$K^{\mathrm{Rubin}}=\begin{bmatrix}(G_{tt}^{R})^{-1} & -(G_{tt}^{R})^{-1}\Phi_{tk}^{N}\\ 0 & I_{kk}\end{bmatrix}^{T}\begin{bmatrix}G_{tt}^{R} & 0\\ 0 & \omega_{k}^{2}\end{bmatrix}\begin{bmatrix}(G_{tt}^{R})^{-1} & -(G_{tt}^{R})^{-1}\Phi_{tk}^{N}\\ 0 & I_{kk}\end{bmatrix}=\begin{bmatrix}(G_{tt}^{R})^{-1} & -(G_{tt}^{R})^{-1}\Phi_{tk}^{N}\\ -(\Phi_{tk}^{N})^{T}(G_{tt}^{R})^{-1} & \omega_{kk}^{2}+(\Phi_{tk}^{N})^{T}(G_{tt}^{R})^{-1}\Phi_{tk}^{N}\end{bmatrix}$$

$$M^{MacNeal}=\begin{bmatrix}0 & 0\\ 0 & I_{kk}\end{bmatrix}$$

Mixed-Interface Methods (Def.)

Physical DoFs in the Modal Reduction Coordinate Transformation may be ALL FIXED, ALL FREE, or ANY Combination of FIXED and FREE when calculating the Normal Modes



Hintz (1975)
Method of Constraint Modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & \Phi_{tk}^{E} \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} q_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Hintz's Method of Constraint Modes

(potential for linear dependencies; special synthesis procedure)

Herting (1978)

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

What is the relationship between Herting's Transformation and Hintz's Method of Constraint Modes Transformation?

Relationship Between Herting and Hintz's methods

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & \Phi_{tk}^{E} \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} \end{bmatrix} \begin{Bmatrix} q_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Hintz's method

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & \Phi_{tk}^{E} \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} \end{bmatrix} \begin{bmatrix} I_{tt} & 0 & -\Phi_{tk}^{E} \\ 0 & I_{rr} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Hintz x elementary transformation

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Modified-Hintz

$$\begin{bmatrix} I_{tt} & 0 & -\Phi_{tk}^{E} \\ 0 & I_{rr} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} \begin{bmatrix} I_{tt} & 0 & \Phi_{tk}^{E} \\ 0 & I_{rr} & 0 \\ 0 & 0 & I_{kk} \end{bmatrix} = I$$

Check for Equivalence
(all information retained)

Then per Equivalence Theorem, Herting's and Hintz's Transformations are Equivalent!

$$\begin{bmatrix} I_{tt} & 0 & \Phi_{tk}^{E} \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} \end{bmatrix}$$

Hintz's Method

Equivalent

$$\begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix}$$

Modified Hintz

$$\begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix}$$

Herting's Method

Hintz (1975)
Method of Attachment Modes

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 & \Phi_{rk}^E \\ \Phi_{br}^R & \Phi_{bb}^A & 0 & \Phi_{bk}^E \\ \Phi_{\mathrm{or}}^R & \Phi_{\mathrm{ob}}^A & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} q_r \\ q_b \\ q_r^A \\ q_k \end{Bmatrix}$$

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 & \Phi_{rk}^E \\ \Phi_{br}^R & \Phi_{bb}^A & 0 & \Phi_{bk}^E \\ \Phi_{\mathrm{or}}^R & \Phi_{\mathrm{ob}}^A & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E \end{bmatrix} \begin{bmatrix} I_{rr} & 0 & 0 & -\Phi_{rk}^E \\ 0 & I_{bb} & 0 & 0 \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_r \\ q_b \\ q_r^A \\ q_k \end{Bmatrix}$$

e1

Hintz
Method of Attachment Modes – Cont'd

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 & 0 \\ \Phi_{br}^{R} & \Phi_{bb}^{A} & 0 & \Phi_{bk}^{E} - \Phi_{br}^{R}\Phi_{rk}^{E} \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{A} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{\mathrm{or}}^{R}\Phi_{rk}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ q_b \\ q_r^{A} \\ q_k \end{Bmatrix}$$

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 & 0 \\ \Phi_{br}^{R} & \Phi_{bb}^{A} & 0 & \Phi_{bk}^{E} - \Phi_{br}^{R}\Phi_{rk}^{E} \\ \Phi_{\mathrm{or}}^{R} & \Phi_{\mathrm{ob}}^{A} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{\mathrm{or}}^{R}\Phi_{rk}^{E} \end{bmatrix} \begin{bmatrix} I_{rr} & 0 & 0 & 0 \\ -\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \left(\Phi_{bb}^{A}\right)^{-1} & 0 & -\left(\Phi_{bb}^{A}\right)^{-1}\left(\Phi_{bk}^{E} - \Phi_{br}^{R}\Phi_{rk}^{E}\right) \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_r^{A} \\ q_k \end{Bmatrix}$$

e2

Hintz
Method of Attachment Modes
Cont'd

$$\begin{bmatrix} I_{rr} & 0 & 0 & -\Phi_{rk}^{E} \\ 0 & I_{bb} & 0 & 0 \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix}\begin{bmatrix} I_{rr} & 0 & 0 & 0 \\ -\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \left(\Phi_{bb}^{A}\right)^{-1} & 0 & -\left(\Phi_{bb}^{A}\right)^{-1}\left(\Phi_{bk}^{E}-\Phi_{br}^{R}\Phi_{rk}^{E}\right) \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix}$$

e1 e2

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 & 0 \\ 0 & I_{bb} & 0 & 0 \\ \Phi_{\mathrm{or}}^{R}-\Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{A}\left(\Phi_{bb}^{A}\right)^{-1} & \Phi_{\mathrm{or}}^{A} & -\Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1}\left(\Phi_{bk}^{E}-\Phi_{br}^{R}\Phi_{rk}^{E}\right)+\Phi_{ok}^{E}-\Phi_{\mathrm{or}}^{R}\Phi_{rk}^{E} \end{bmatrix}\begin{Bmatrix} x_r \\ x_b \\ q_r^{A} \\ q_k \end{Bmatrix}$$

Comparison of Hintz Method of Constraint Modes and Attachment Modes

$$\begin{Bmatrix} x_t \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{tt} & 0 & 0 \\ \Phi_{ot}^{C} & \Phi_{\mathrm{or}}^{A} & \Phi_{ok}^{E} - \Phi_{ot}^{C}\Phi_{tk}^{E} \end{bmatrix} \begin{Bmatrix} x_t \\ q_r^A \\ q_k \end{Bmatrix}$$

Modified Hintz Method of Constraint Modes

$$\begin{Bmatrix} x_r \\ x_b \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{rr} & 0 & 0 & 0 \\ 0 & I_{bb} & 0 & 0 \\ \Phi_{\mathrm{or}}^{R} - \Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1}\Phi_{br}^{R} & \Phi_{\mathrm{ob}}^{A}\left(\Phi_{bb}^{A}\right)^{-1} & \Phi_{\mathrm{or}}^{A} & -\Phi_{ob}^{A}\left(\Phi_{bb}^{A}\right)^{-1}\left(\Phi_{bk}^{E} - \Phi_{br}^{R}\Phi_{rk}^{E}\right) + \Phi_{ok}^{E} - \Phi_{\mathrm{or}}^{R}\Phi_{rk}^{E} \end{bmatrix} \begin{Bmatrix} x_r \\ x_b \\ q_r^A \\ q_k \end{Bmatrix}$$

Modified Hintz Method of Attachment Modes

Residual Flexibility Mixed-Boundary (RFMB)*

$$\begin{Bmatrix} x_j \\ x_c \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ \Phi_{cj}^C & G_{cc}^R & 0 & \Phi_{ck}^E \\ \Phi_{\mathrm{oj}}^C & G_{\mathrm{oc}}^R & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E \end{bmatrix} \begin{Bmatrix} x_j \\ q_c \\ q_r^A \\ q_k \end{Bmatrix}$$

RFMB Component Mode Sets

$$\begin{Bmatrix} x_j \\ x_c \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ \Phi_{cj}^C & G_{cc}^R & 0 & \Phi_{ck}^E \\ \Phi_{\mathrm{oj}}^C & G_{\mathrm{oc}}^R & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E \end{bmatrix} \begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ -\left(G_{cc}^R\right)^{-1}\Phi_{cj}^C & \left(G_{cc}^R\right)^{-1} & 0 & -\left(G_{cc}^R\right)^{-1}\Phi_{ck}^E \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix} \begin{Bmatrix} x_j \\ x_c \\ q_r^A \\ q_k \end{Bmatrix}$$

RFMB x Elementary transformation

$$\begin{Bmatrix} x_j \\ x_c \\ x_o \end{Bmatrix} = \begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ 0 & I_{cc} & 0 & 0 \\ \Phi_{\mathrm{oj}}^C - G_{oc}^R\left(G_{cc}^R\right)^{-1}\Phi_{cj}^C & G_{\mathrm{oc}}^R\left(G_{cc}^R\right)^{-1} & \Phi_{\mathrm{or}}^A & \Phi_{ok}^E - G_{oc}^R\left(G_{cc}^R\right)^{-1}\Phi_{ck}^E \end{bmatrix} \begin{Bmatrix} x_j \\ x_c \\ q_r^A \\ q_k \end{Bmatrix}$$

RFMB

Recall J = R + B

* Majed & Henkel ASD, 2000

RFMB (Cont'd)

$$\begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ -\left(G_{cc}^{R}\right)^{-1}\Phi_{cj}^{C} & \left(G_{cc}^{R}\right)^{-1} & 0 & -\left(G_{cc}^{R}\right)^{-1}\Phi_{ck}^{E} \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix}\begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ \Phi_{cj}^{C} & G_{cc}^{R} & 0 & \Phi_{ck}^{E} \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix} = I$$

Equivalence Theorem (information conservation)

$$K^{\mathrm{RFMB}} = \begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ -\left(G_{cc}^{R}\right)^{-1}\Phi_{cj}^{C} & \left(G_{cc}^{R}\right)^{-1} & 0 & -\left(G_{cc}^{R}\right)^{-1}\Phi_{ck}^{E} \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix}^{T}\begin{bmatrix} K_{jj}^{C} & 0 & 0 & 0 \\ 0 & G_{cc}^{R} & 0 & 0 \\ 0 & 0 & K_{rr}^{A} & 0 \\ 0 & 0 & 0 & \omega_k^2 \end{bmatrix}\begin{bmatrix} I_{jj} & 0 & 0 & 0 \\ -\left(G_{cc}^{R}\right)^{-1}\Phi_{cj}^{C} & \left(G_{cc}^{R}\right)^{-1} & 0 & -\left(G_{cc}^{R}\right)^{-1}\Phi_{ck}^{E} \\ 0 & 0 & I_{rr} & 0 \\ 0 & 0 & 0 & I_{kk} \end{bmatrix}$$

Concluding Remarks

- A simple systematic approach based on elementary matrix transformations was utilized to derive and study a large number of fixed, free, and mixed-interface modal reduction methods
- These elementary transformations were utilized to derive “modified” forms, study linear dependence, circumvent special synthesis procedures, and show “equivalence” between different forms
- It was shown that although certain forms are mathematically equivalent, they are still superior due to ease of direct derivation from test data
- Special topics involving damping and changing modal boundary conditions were addressed

Questions???

Thank you for Attending…